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Nonlinear evolution equations connected with the matrix Schrödinger spectral problem

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Abstract. In this paper, a generalized 2×2 Schrödinger spectral problem and $N\times N$ reduced matrix Schrödinger spectral problem are considered. The corresponding hierarchies of nonlinear evolution equations are derived.

1. Introduction

A central and very active topic in theory of integrable systems is to search for as many new integrable systems as possible. During the past twenty years or so, much attention has been paid to the following spectral problem (see [1–11] and references therein):

$$\psi_x = U\psi \tag{1}$$

where $U = e_0(\lambda) + u_1 e_1(\lambda) + \cdots + u_p e_p(\lambda)$, $e_i(\lambda)$ $(i = 0, 1, \cdots, p)$ belongs to a loop algebra $G\otimes C[\lambda, \lambda^{-1}]$, and u_i $(i = 1, \ldots, p)$ is taken from the Schwartz space $S(-\infty, \infty)$. Compared with (1), the study of other kinds of spectral problems has received relatively little attention. In [12], Calogero and Degasperis considered the following matrix Schrödinger spectral problem:

$$\psi_{xx} = (\lambda I + U)\psi \tag{2}$$

where I is a $N\times N$ unit matrix and U is a potential matrix. In [13], Levi considered the following matrix spectral problem with a non-diagonal eigenvalue:

$$\psi_{xx} = \begin{bmatrix} u+\phi & \lambda \\ \lambda & u-\phi \end{bmatrix}\psi \tag{3}$$

and derived a hierarchy of coupled KdV equations which contains among its members the Hirota–Satsuma equation

$$u_t = -\tfrac{1}{4}(-u_{xxx} + 6uu_x - 12\phi\phi_x)$$
$$\phi_t = -\tfrac{1}{2}(\phi_{xxx} - 3u\phi_x).$$

In this paper, we shall consider two spectral problems. In section 2, a generalized 2×2 Schrödinger spectral problem and the corressponding hierarchy of integrable systems are presented. A hierarchy of nonlinear evolution equations connected with the $N\times N$ reduced matrix Schrödinger spectral problem is derived in section 3. Finally, concluding remarks are given in section 4.

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2. A generalized 2×2 Schrödinger spectral problem and the corressponding integrable equations

A generalized 2 × 2 Schrödinger spectral problem under consideration is

$$\psi_{xx} = U\psi = \begin{bmatrix} \lambda\phi + s & u \\ v & \lambda\phi + s \end{bmatrix} \psi \tag{4}$$

where u, v, s, $\phi - \phi_0 \in S(-\infty, \infty)$, ϕ_0 is a non-zero constant, and $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$. In order to derive nonlinear evolution equations connected with (4), we impose an auxiliary spectral problem on ψ, of the form

$$\psi_t = V\psi + W\psi_x. \tag{5}$$

By requiring the compatibility of equation (5) with equation (4), we obtain the following two matrix equations [13]:

$$U_t = V_{xx} + 2W_x U + WU_x + [V, U] \tag{6a}$$

$$W_{xx} + 2V_x + [W, U] = 0 \tag{6b}$$

We now derive nonlinear evolution equations connected with (4) via the following steps.

First we consider the stationary equations of (6),

$$V_{xx} + 2W_x U + WU_x + [V, U] = 0 \tag{7a}$$

$$W_{xx} + 2V_x + [W, U] = 0. \tag{7b}$$

We take

$$V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad W = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{a} \end{bmatrix}.$$

Then (7) becomes

$$\bar{a}_{xx} + 2a_x + v\bar{b} - u\bar{c} = 0 \tag{8a}$$

$$\bar{a}_{xx} + 2d_x + u\bar{c} - v\bar{b} = 0 \tag{8b}$$

$$\bar{b}_x + 2b = 0 \tag{8c}$$

$$\bar{c}_x + 2c = 0 \tag{8d}$$

$$a_{xx} + 2\bar{a}_x(\lambda\phi + s) + 2v\bar{b}_x + \bar{a}(\lambda\phi_x + s_x) + v_x\bar{b} + vb - uc = 0 \tag{8e}$$

$$d_{xx} + 2\bar{a}_x(\lambda\phi + s) + 2u\bar{c}_x + \bar{a}(\lambda\phi_x + s_x) + u_x\bar{c} + uc - vb = 0 \tag{8f}$$

$$b_{xx} + 2\bar{b}_x(\lambda\phi + s) + 2u\bar{a}_x + \bar{b}(\lambda\phi_x + s_x) + u_x\bar{a} + u(a - d) = 0 \tag{8g}$$

$$c_{xx} + 2\bar{c}_x(\lambda\phi + s) + 2v\bar{a}_x + \bar{c}(\lambda\phi_x + s_x) + v_x\bar{a} + v(d - a) = 0. \tag{8h}$$

Notice that (8) is an overdetermined system of equations with respect to a, b, c, d, $\bar{a}$, $\bar{b}$ and $\bar{c}$. We can easily verify that (8) is compatible. In fact, it follows from (8a) and (8b) that

$$(a-d)_{xx}+(v\bar{b}-u\bar{c})_x=0. \tag{9}$$

By the use of (8f), (8c), (8d) and (9), we have

$$\begin{aligned}a_{xx}+2\bar{a}_x(\lambda\phi+s)+2v\bar{b}_x+\bar{a}(\lambda\phi_x+s_x)+v_x\bar{b}+vb-uc\\&=(a-d)_{xx}+2v\bar{b}_x+v_x\bar{b}+2(vb-uc)-2u\bar{c}_x-u_x\bar{c}\\&=(a-d)_{xx}+(v\bar{b}-u\bar{c})_x\\&=0\end{aligned}$$

which means that (8e) can be deduced from (8a)–(8d) and (8f), i.e. (8) is compatible. Substitution of the expressions

$$\begin{bmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{a}\end{bmatrix}=\sum_{m=0}^{\infty}\begin{bmatrix}\bar{a}_m & \bar{b}_m\\ \bar{c}_m & \bar{a}_m\end{bmatrix}\lambda^{-m} \qquad \begin{bmatrix}a & b\\ c & d\end{bmatrix}=\sum_{m=0}^{\infty}\begin{bmatrix}a_m & b_m\\ c_m & d_m\end{bmatrix}\lambda^{-m}$$

into (8) yields

$$\begin{aligned}&\bar{a}_{m_{xx}}+2a_{m_x}+v\bar{b}_m-u\bar{c}_m=0\\&\bar{a}_{m_{xx}}+2d_{m_x}+u\bar{c}_m-v\bar{b}_m=0\\&\bar{b}_{m_x}+2b_m=0\\&\bar{c}_{m_x}+2c_m=0\\&a_{m_{xx}}+2\phi\bar{a}_{m+1_x}+2s\bar{a}_{m_x}+2v\bar{b}_{m_x}+\phi_x\bar{a}_{m+1}+s_x\bar{a}_m+v_x\bar{b}_m+vb_m-uc_m=0\\&d_{m_{xx}}+2\phi\bar{a}_{m+1_x}+2s\bar{a}_{m_x}+2u\bar{c}_{m_x}+\phi_x\bar{a}_{m+1}+s_x\bar{a}_m+u_x\bar{c}_m+uc_m-vb_m=0\\&b_{m_{xx}}+2\phi\bar{b}_{m+1_x}+2s\bar{b}_{m_x}+2u\bar{a}_{m_x}+\phi_x\bar{b}_{m+1}+s_x\bar{b}_m+u_x\bar{a}_m+u(a_m-d_m)=0\\&c_{m_{xx}}+2\phi\bar{c}_{m+1_x}+2s\bar{c}_{m_x}+2v\bar{a}_{m_x}+\phi_x\bar{c}_{m+1}+s_x\bar{c}_m+v_x\bar{a}_m+v(d_m-a_m)=0.\end{aligned} \tag{10}$$

In order to fix the integral constant arising from calculation, we define the rank for $\partial/\partial x$, u, v, s and ϕ as follows [7, 8]:

$$\begin{aligned}&\mathrm{rank}\left(\frac{\partial}{\partial x}\right)=1\\&\mathrm{rank}(u)=\mathrm{rank}(v)=\mathrm{rank}(s)=2\\&\mathrm{rank}(\phi)=0\end{aligned}$$

and follow the homogeneous rank convention: both sides of an equation have the same rank.

We now give the first few of $\bar{a}_m$, $\bar{b}_m$, $\bar{c}_m$, a_m, b_m, c_m and d_m in two cases.

Case a

$$\bar{a}_0 = \alpha_1 \phi^{-1/2} \qquad \bar{b}_0 = \beta_1 \phi^{-1/2} \qquad \bar{c}_0 = \gamma_1 \phi^{-1/2}$$

$$b_0 = \tfrac{1}{4}\beta_1 \phi^{-3/2}\phi_x \qquad c_0 = \tfrac{1}{4}\gamma_1 \phi^{-3/2}\phi_x$$

$$a_0 = \tfrac{1}{4}\alpha_1 \phi^{-3/2}\phi_x + \tfrac{1}{8}\partial^{-1}[(\gamma_1 u - \beta_1 v)\phi^{-3/2}\phi_x]$$

$$d_0 = \tfrac{1}{4}\alpha_1 \phi^{-3/2}\phi_x - \tfrac{1}{8}\partial^{-1}[(\gamma_1 u - \beta_1 v)\phi^{-3/2}\phi_x], \ldots$$

where α_1, β_1 and γ_1 are constants and $|\alpha_1|^2 + |\beta_1|^2 + |\gamma_1|^2 \neq 0$.

Case b

$$\bar{a}_0 = \bar{b}_0 = \bar{c}_0 = b_0 = c_0 = 0$$

$$a_0 = \alpha_2 = \text{constant} \qquad d_0 = \beta_2 = \text{constant} \qquad |\alpha_2|^2 + |\beta_2|^2 \neq 0$$

$$\bar{a}_1 = 0 \qquad \bar{b}_1 = \tfrac{1}{2}(\beta_2 - \alpha_2)\phi^{-1/2}\partial^{-1}(\phi^{-1/2}u)$$

$$\bar{c}_1 = \tfrac{1}{2}(\alpha_2 - \beta_2)\phi^{-1/2}\partial^{-1}(\phi^{-1/2}v), \ldots .$$

In general, we have the following recursion relation:

$$(\bar{a}_{m+1}, \bar{b}_{m+1}, \bar{c}_{m+1})^{\mathrm{T}} = L(\bar{a}_m, \bar{b}_m, \bar{c}_m)^{\mathrm{T}} \tag{11}$$

where $L = (L_{ij})_{1 \leqslant i,j \leqslant 3}$ and

$$L_{11} = \tfrac{1}{2}\phi^{-1/2}\partial^{-1}\phi^{-1/2}(\tfrac{1}{2}\partial^3 - 2s\partial - s_x) \qquad L_{12} = \tfrac{1}{2}\phi^{-1/2}\partial^{-1}\phi^{-1/2}(-\tfrac{1}{2}v_x - v\partial)$$

$$L_{13} = \tfrac{1}{2}\phi^{-1/2}\partial^{-1}\phi^{-1/2}(-u\partial - \tfrac{1}{2}u_x) \qquad L_{21} = \tfrac{1}{2}\phi^{-1/2}\partial^{-1}\phi^{-1/2}(-2u\partial - u_x)$$

$$L_{22} = \tfrac{1}{2}\phi^{-1/2}\partial^{-1}\phi^{-1/2}(\tfrac{1}{2}\partial^3 - 2s\partial - s_x + u\partial^{-1}v) \qquad L_{23} = \tfrac{1}{2}\phi^{-1/2}\partial^{-1}\phi^{-1/2}(-u\partial^{-1}u)$$

$$L_{31} = \tfrac{1}{2}\phi^{-1/2}\partial^{-1}\phi^{-1/2}(-2v\partial - v_x) \qquad L_{32} = \tfrac{1}{2}\phi^{-1/2}\partial^{-1}\phi^{-1/2}(-v\partial^{-1}v)$$

$$L_{33} = \tfrac{1}{2}\phi^{-1/2}\partial^{-1}\phi^{-1/2}(\tfrac{1}{2}\partial^3 - 2s\partial - s_x + v\partial^{-1}u).$$

Secondly we set

$$\Delta_{1n} = -\frac{1}{2}\begin{bmatrix} \delta_{nx} & 0 \\ 0 & \delta_{nx} \end{bmatrix} \qquad \Delta_{2n} = \begin{bmatrix} \delta_n & 0 \\ 0 & \delta_n \end{bmatrix}$$

$$V^{(n)} = (\lambda^n V)_+ + \Delta_{1n} \equiv \sum_{i=0}^{n} \lambda^{n-i} \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} + \Delta_{1n}$$

$$W^{(n)} = (\lambda^n W)_+ + \Delta_{2n} \equiv \sum_{i=0}^{n} \lambda^{n-i} \begin{bmatrix} \bar{a}_i & \bar{b}_i \\ \bar{c}_i & \bar{a}_i \end{bmatrix} + \Delta_{2n}$$

where δ_n is an arbitrary function. Then, from (4) and $\psi_{t_n} = V^{(n)}\psi + W^{(n)}\psi_x$, we can deduce a hierarchy of equations

$$
\begin{aligned}
\phi_{t_n} &= 2\phi\delta_{n_x} + \phi_x\delta_n \\
s_{t_n} &= -2\phi\bar{a}_{n+1_x} - \phi_x\bar{a}_{n+1} - \tfrac{1}{2}\delta_{n_{xxx}} + 2s\delta_{n_x} + s_x\delta_n \\
u_{t_n} &= -2\phi\bar{b}_{n+1_x} - \phi_x\bar{b}_{n+1} + 2u\delta_{n_x} + u_x\delta_n \\
v_{t_n} &= -2\phi\bar{c}_{n+1_x} - \phi_x\bar{c}_{n+1} + 2v\delta_{n_x} + v_x\delta_n
\end{aligned} \tag{12}
$$

where $\bar{a}_{n+1}$, $\bar{b}_{n+1}$ and $\bar{c}_{n+1}$ are given by recursion relations (10). In particular, if we choose $\phi = 1, \delta_n = 0$, then we have two hierarchies of equations respectively corresponding to two different choices of $\bar{a}_0$, $\bar{b}_0$, $\bar{c}_0$, a_0, b_0, c_0 and d_0.

The first hierarchy is

$$(s,u,v)^{\mathrm{T}}_{t_n} = J\tilde{L}^{n+1}(\alpha_1,\beta_1,\gamma_1)^{\mathrm{T}}.$$

The second hierarchy is

$$(s,u,v)^{\mathrm{T}}_{t_n} = J\tilde{L}^{n}(0, \tfrac{1}{2}(\beta_2-\alpha_2)\partial^{-1}u, \tfrac{1}{2}(\alpha_2-\beta_2)\partial^{-1}v)^{\mathrm{T}}$$

where

$$
J = \begin{bmatrix} -2\partial & 0 & 0 \\ 0 & -2\partial & 0 \\ 0 & 0 & -2\partial \end{bmatrix}
$$

$$
\tilde{L} = \begin{bmatrix}
\frac{1}{4}\partial^2 - \frac{1}{2}\partial^{-1}s\partial - \frac{1}{2}s & -\frac{1}{4}v - \frac{1}{4}\partial^{-1}v\partial & -\frac{1}{4}u - \frac{1}{4}\partial^{-1}u\partial \\
-\frac{1}{2}u - \frac{1}{2}\partial^{-1}u\partial & \frac{1}{4}\partial^2 - \frac{1}{2}s - \frac{1}{2}\partial^{-1}s\partial + \frac{1}{2}\partial^{-1}u\partial^{-1}v & -\frac{1}{2}\partial^{-1}u\partial^{-1}u \\
-\frac{1}{2}v - \frac{1}{2}\partial^{-1}v\partial & -\frac{1}{2}\partial^{-1}v\partial^{-1}v & \frac{1}{4}\partial^2 - \frac{1}{2}s - \frac{1}{2}\partial^{-1}s\partial + \frac{1}{2}\partial^{-1}v\partial^{-1}u
\end{bmatrix}.
$$

Remark. The spectral problem (4) with $\phi = 1$ is just a 2×2 reduced Schrödinger spectral problem. Note that the ϕ in (4) guarantees that the nonlinear evolution equations (NLEE) (12) connected with (4) contain an arbitrary function δ_n which also appears in the NLEE connected with the Giachetti–Johnson spectral problem [8].

3. Nonlinear evolution equations connected with the $N \times N$ reduced Schrödinger spectral problem

In this section, we shall consider the reduction problem of (2). It is easy to see that the number of potential functions increases quadratically with N in (2). It is of practical importance to find reduced spectral problems where the number of potential functions is considerably less than N^2. In the following, a number of such reduced systems will be presented. To this end, we set e_i $(i = 0, \ldots, n)$ as an $N \times N$ matrix, and $e_0 = I$ as a unit matrix. Further we assume that e_i $(i = 0, \ldots, n)$ is linearly independent and the linear span of $\{e_i\}_{i=0}^{n}$ is closed under matrix multiplication, i.e.

$$e_i \cdot e_j = \sum_{k=0}^{n} C_{ij}^{k} e_k \tag{13}$$

where c_{ij}^k is a constant. We consider the following $N \times N$ reduced Schrödinger spectral problem:

$$\psi_{xx} = U\psi$$

$$U = \lambda e_0 + \sum_{i=0}^{n} u_i e_i \tag{14}$$

where $u_i \in S(-\infty, \infty)$ $(i = 0, \ldots, n)$. In order to derive nonlinear evolution equations connected with (14), we impose an auxiliary spectral problem on ψ as in section 2,

$$\psi_t = V\psi + W\psi_x. \tag{15}$$

By requiring the compatibility of equation (15) with equation (14), we get the following two matrix equations:

$$U_t = V_{xx} + 2W_x U + WU_x + [V, U] \tag{16a}$$

$$W_{xx} + 2V_x + [W, U] = 0. \tag{16b}$$

We now derive nonlinear evolution equations connected with (14) via the following steps.

First we consider the stationary equations of (16)

$$V_{xx} + 2W_x U + WU_x + [V, U] = 0 \tag{17a}$$

$$W_{xx} + 2V_x + [W, U] = 0. \tag{17b}$$

Set

$$V = \sum_{i=0}^{n} a_i e_i \qquad W = \sum_{i=0}^{n} \bar{a}_i e_i.$$

Then we deduce from (17) that

$$\bar{a}_{k_{xx}} + 2a_{k_x} + \sum_{i=1}^{n}\sum_{j=1}^{n} \bar{a}_i u_j (C_{ij}^k - C_{ji}^k) = 0 \qquad k = 0, 1, \ldots, n \tag{18}$$

$$a_{k_{xx}} + 2(\lambda + u_0)\bar{a}_{k_x} + 2\sum_{i=0}^{n}\sum_{j=1}^{n} \bar{a}_{i_x} u_j C_{ij}^k + \sum_{i=0}^{n}\sum_{j=0}^{n} \bar{a}_i u_{j_x} C_{ij}^k$$

$$+ \sum_{i=1}^{n}\sum_{j=1}^{n} a_i u_j (C_{ij}^k - C_{ji}^k) = 0 \qquad k = 0, 1, \ldots, n. \tag{19}$$

Substitution of

$$a_k = \sum_{m=0}^{\infty} a_k^{(m)} \lambda^{-m} \qquad \bar{a}_k = \sum_{m=0}^{\infty} \bar{a}_k^{(m)} \lambda^{-m}$$

into (18) and (19) yields

$$a_{k_x}^{(m)} = -\tfrac{1}{2}\bar{a}_{k_{xx}}^{(m)} - \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} u_j (C_{ij}^k - C_{ji}^k)\bar{a}_i^{(m)} \tag{20}$$

$$\bar{a}_{k_x}^{(m+1)} = -\tfrac{1}{2}a_{k_{xx}}^{(m)} - u_0\bar{a}_{k_x}^{(m)} - \sum_{i=0}^{n}\sum_{j=1}^{n} u_j C_{ij}^k \bar{a}_{i_x}^{(m)}$$
$$- \tfrac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n} u_{j_x} C_{ij}^k \bar{a}_i^{(m)} - \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} u_j (C_{ij}^k - C_{ji}^k) a_i^{(m)}. \tag{21}$$

We now begin the recursive process with two different choices of $\bar{a}_k^{(0)}$ and $a_k^{(0)}$ $(k = 0, \ldots, n)$.

Case a

$$\bar{a}_k^{(0)} = \alpha_k = \text{constant} \qquad k = 0, \ldots, n \qquad \sum_{k=0}^{n} |\alpha_k|^2 \neq 0$$

$$a_k^{(0)} = -\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} (C_{ij}^k - C_{ji}^k)\alpha_i \partial^{-1} u_j.$$

Case b

$$\bar{a}_k^{(0)} = 0 \qquad a_k^{(0)} = \beta_k = \text{constant} \qquad k = 0, \ldots, n \qquad \sum_{k=0}^{n} |\beta_k|^2 \neq 0.$$

In general, we have from (20) and (21)

$$(2\bar{a}_{0_x}^{(m+1)}, \ldots, 2\bar{a}_{n_x}^{(m+1)})^{\mathrm{T}} = L(2\bar{a}_{0_x}^{(m)}, \ldots, 2\bar{a}_{n_x}^{(m)})^{\mathrm{T}}$$

where $L = (L_{kl})_{0 \leqslant k,l \leqslant n}$, and

$$L_{kl} = \tfrac{1}{4}\delta_{kl}\partial^2 - \tfrac{1}{4}(1 - \delta_{l0})\sum_{j=1}^{n} (C_{jl}^k - C_{lj}^k)\partial u_j \partial^{-1}$$
$$- \delta_{kl}u_0 - \sum_{j=1}^{n} C_{lj}^k u_j - \tfrac{1}{2}\sum_{j=0}^{n} C_{lj}^k u_{j_x}\partial^{-1}$$
$$+ \tfrac{1}{4}(1 - \delta_{l0})\sum_{j=1}^{n} (C_{lj}^k - C_{jl}^k) u_j$$
$$- \tfrac{1}{4}(1 - \delta_{l0})\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{n} (C_{ij}^k - C_{ji}^k)(C_{rl}^i - C_{lr}^i) u_j \partial^{-1} u_r \partial^{-1}$$

$$\delta_{kl} = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}.$$

Set

$$V^{(m)} = (\lambda^m V)_+ \equiv \sum_{i=0}^{m}\sum_{k=0}^{n} \lambda^{m-i} a_k^{(i)} e_k$$

$$W^{(m)} = (\lambda^m W)_+ \equiv \sum_{i=0}^{m}\sum_{k=0}^{n} \lambda^{m-i} \bar{a}_k^{(i)} e_k.$$

Then from (14) and $\psi_{t_m} = V^{(m)}\psi + W^{(m)}\psi_x$, we can deduce a hierarchy of equations

$$(u_0, \ldots, u_n)_{t_m} = -(2\bar{a}_{0_x}^{(m+1)}, \ldots, 2\bar{a}_{n_x}^{(m+1)})$$

where $\bar{a}_k^{(m+1)}$ $(k = 0, \ldots, n)$ is given by (18) and (19). Corresponding to two different choices of $a_k^{(0)}$ and $\bar{a}_k^{(0)}$ $(k = 0, \ldots, n)$, we have two hierarchies of equations:

$$(u_0, \ldots, u_n)_{t_m}^{\mathrm{T}} = L^m \Bigg(\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}(C_{ij}^0 + C_{ji}^0)\alpha_i u_{j_x} - \frac{1}{2}\sum_{1\leqslant i,j,l,p\leqslant n}(C_{ij}^0 - C_{ji}^0)(C_{lp}^i - C_{pl}^i)\alpha_l u_j \partial^{-1} u_p,$$
$$\ldots, \frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}(C_{ij}^n + C_{ji}^n)\alpha_i u_{j_x} - \frac{1}{2}\sum_{1\leqslant i,j,l,p\leqslant n}(C_{ij}^n - C_{ji}^n)(C_{lp}^i - C_{pl}^i)\alpha_l u_j \partial^{-1} u_p \Bigg)^{\mathrm{T}} \quad (22)$$

$$(u_0, \ldots, u_n)_{t_m}^{\mathrm{T}} = L^m \left(\sum_{i=1}^{n}\sum_{j=1}^{n}(C_{ij}^0 - C_{ji}^0)\beta_i u_j, \ldots, \sum_{i=1}^{n}\sum_{j=1}^{n}(C_{ij}^n - C_{ji}^n)\beta_i u_j \right)^{\mathrm{T}}. \quad (23)$$

Note that the spectral problem (2) is a special case of (14). Therefore, we can obtain the corresponding integrable hierarchies connected with (2) from (22) and (23). In the following, we give another example as a concrete application of the obtained results.

Choose

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The spectral problem in this case is

$$\psi_{xx} = \begin{bmatrix} \lambda + u_0 + u_1 & 0 \\ u_2 & \lambda + u_0 \end{bmatrix} \psi. \quad (24)$$

We have

$$C_{0i}^0 = C_{i0}^0 = \begin{cases} 1 & i = 0 \\ 0 & i = 1, 2 \end{cases}$$

$$C_{0i}^1 = C_{i0}^1 = \begin{cases} 0 & i = 0, 2 \\ 1 & i = 1 \end{cases}$$

$$C_{0i}^2 = C_{i0}^2 = \begin{cases} 0 & i = 0, 1 \\ 1 & i = 2 \end{cases}$$

$$C_{22}^i = 0 \qquad i = 0, 1, 2$$

$$C_{11}^0 = C_{11}^2 = C_{12}^2 = 0$$

$$C_{12}^0 = C_{21}^0 = C_{12}^1 = C_{21}^1 = 0$$

$$C_{11}^1 = C_{21}^2 = 1. \quad (25)$$

The corresponding hierarchies of nonlinear evolution equations connected with (24) are (22) and (23) with (25), which can be merged into the following hierarchy:

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}_{t_m} = \bar{L}^m \begin{pmatrix} \alpha_0 u_{0x} \\ \alpha_0 u_{1x} + \alpha_1 u_{0x} + \alpha_1 u_{1x} \\ \alpha_2 u_{0x} + \frac{1}{2}\alpha_2 u_{1x} + (\alpha_0 + \frac{1}{2}\alpha_1) u_{2x} \\ + \frac{1}{2} u_1 \partial^{-1}(\alpha_1 u_2 - \alpha_2 u_1) + \beta_2 u_1 - \beta_1 u_2 \end{pmatrix} \tag{26}$$

where $\bar{L} = (L_{ij})_{0 \leqslant i,j \leqslant 2}$ and

$$L_{00} = \tfrac{1}{4}\partial^2 - u_0 - \tfrac{1}{2}u_{0x}\partial^{-1} \qquad L_{01} = L_{02} = 0$$

$$L_{10} = -u_1 - \tfrac{1}{2}u_{1x}\partial^{-1} \qquad L_{11} = \tfrac{1}{4}\partial^2 - u_0 - u_1 - \tfrac{1}{2}u_{0x}\partial^{-1} - \tfrac{1}{2}u_{1x}\partial^{-1}$$

$$L_{12} = 0 \qquad L_{20} = -u_2 - \tfrac{1}{2}u_{2x}\partial^{-1}$$

$$L_{21} = -\tfrac{1}{4}\partial u_2 \partial^{-1} - \tfrac{1}{4}u_2 - \tfrac{1}{4}u_1\partial^{-1}u_2\partial^{-1}$$

$$L_{22} = \tfrac{1}{4}\partial^2 - u_0 - \tfrac{1}{2}u_{0x}\partial^{-1} - \tfrac{1}{4}u_{1x}\partial^{-1} - \tfrac{1}{2}u_1 + \tfrac{1}{4}u_1\partial^{-1}u_1\partial^{-1}.$$

4. Concluding remarks

In this paper, we have derived the nonlinear evolution equations connected with the matrix Schrödinger spectral problem. It is of both theoretical and practical value to find as many new integrable systems as possible and to elucidate in depth their algebraic and geometric properties. In theory, it will greatly help to formulate a criterion for integrability, which is a long-standing open problem; and in practice it will provide us with dozens of nonlinear evolution equations, which are of potential value in physical applications. Such an example is that integrable equations derived by Wadati *et al* [9] have been found to describe transverse oscillations of elastic beams [14]. Therefore, it is possible that the evolution equations derived in this paper will find physical applications. In addition, the algebraic and geometric properties of these new equations should be considered, which still remains to be done in the future.

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